# Group theoretical methods in Machine Learning

Risi Kondor
Columbia University

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Recall that under translation $f \mapsto f^z$ where

$$f^z(x) = f(z^{-1}x)$$

Goal: construct invariant functionals satisfying

$$q(f^z) = q(z)$$

Generalize to Lie groups and homogeneous spaces

# Lie groups

"A group $G$ is a Lie group if it is also a differentiable manifold on which the maps $x \mapsto x^{-1}$ and $(x, y) \mapsto xy$ are differentiable."

1. For any $g \in G$ and $b \in \mathbb{Z}$ there is a $g^{1/b}$ satisfying $(g^{1/b})^b = g$.

2. Extend to real exponents $g^t$ by continuity and $g^{-t} = (g^t)^{-1}$.

3. $\gamma(t) = g^t$ is a differentiable curve on the manifold and $\gamma(0) = 1_G$ .

4. $\{ \gamma(t) \mid t \in \mathcal{R} \}$ is a subgroup and $\gamma(s)\,\gamma(t) = \gamma(s+t)$.

5. $T_g = \frac{d}{dt}\,\gamma(t)|_{1_G}$ is well defined and

$$\gamma(t) = (\gamma(\delta))^{t/\delta} = (1_G + \delta T)^{t/\delta} = e^{tT} \qquad \delta \to 0.$$

6. The tangent space has a basis $\{T_1, T_2, \ldots, T_n\}$ so

$$g = \exp\left(\sum_{i=1}^{n} \alpha_i\, T_i\right) \qquad \forall\, g \in G.$$

7. The generators $\{T_1, T_2, \ldots, T_n\}$ form an algebra.

8. Locally the structure of $G$ is determined by the commutators $[T_i, T_j] = T_i T_j - T_j T_i$ because

$$\exp(u)\exp(v) = \exp(u + v + \tfrac{1}{2}[u,v] + \tfrac{1}{12}[[u,v],v] - \ldots).$$

# Example

How about when $G$ acts on a space $\mathcal{X}$ and $f : \mathcal{X} \to \mathbb{C}$ ?

$\mathcal{X}$ is a homogeneous space of $G$ if the orbit $\{ g(x_0) \mid g \in G \}$ sweeps out $\mathcal{X}$.

In this case the translate of is defined

$$f^z(g(x_0)) = f((z^{-1}g)(x_0)).$$

# Example

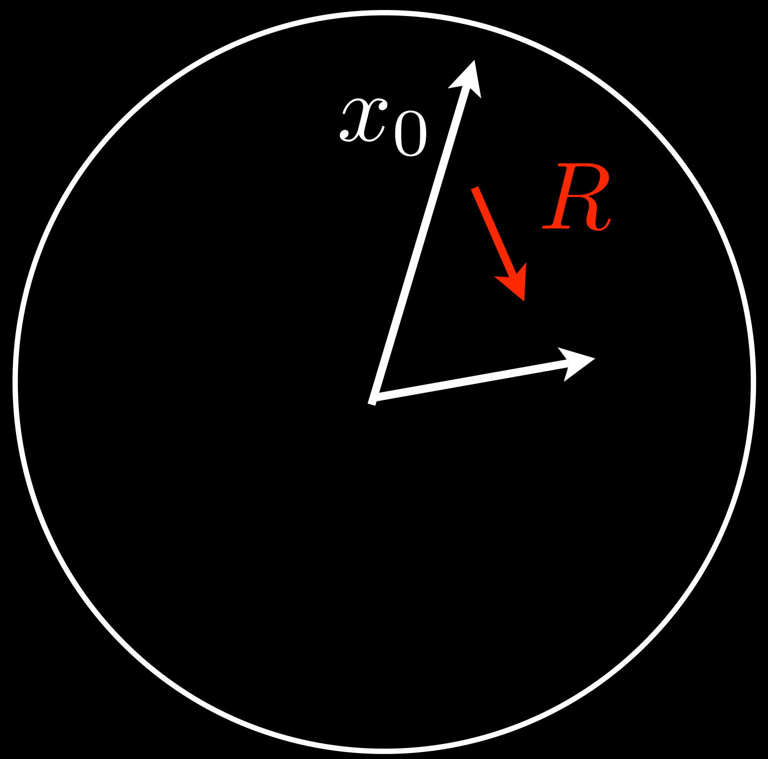
$\mathrm{SO}(3)$ acts on $S_2$ by

$$x \mapsto R\,x$$

If $x_0 = (0,0,1)$ then

$$\{\, R x_0 \mid R \in \mathrm{SO}(3) \,\}$$

sweeps out the entire sphere.



$f$ also induces a function $f\uparrow^G(g) = f(g(x_0))$.

$$f^z\uparrow^G(g) = f(z^{-1}g(x_0)) = f\uparrow^G(z^{-1}g) = \left(f\uparrow^G\right)^z(g)$$

In particular,

$$\widehat{f^z\uparrow^G}(\rho) = \rho(z)\,\widehat{f\uparrow^G}(\rho)$$

So what does $f\uparrow^G$ look like?

The stabilizer $H = \{ g \in G \mid g(x_0) = x_0 \}$ is a subgroup of $G$.

$$(gh)(x_0) = g(h(x_0)) = g(x_0) \qquad \forall\, h \in H$$

$$\mathcal{X} \quad \leftrightarrow \quad \text{set of cosets} \;\; gH = \{ gh \mid h \in H \}$$

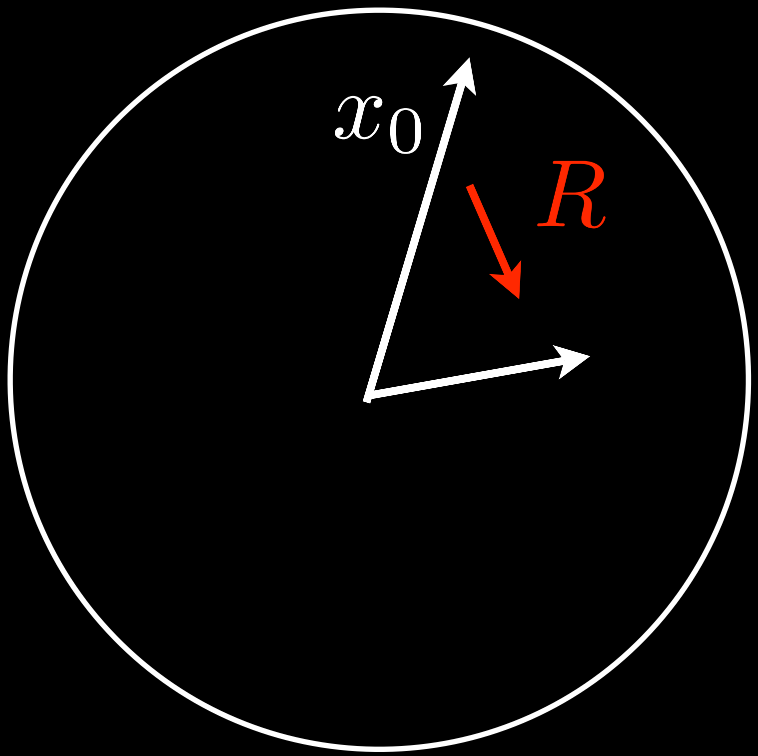
$f \uparrow^G$ is constant on each $gH$ coset.

# Example

$G = \mathrm{SO}(3)$

$x_0 = (0, 0, 1)$

$S_2 = \mathrm{SO}(3)/\mathrm{SO}(2)$




$$x \quad \longleftrightarrow \quad R \cdot \mathrm{SO}(2)$$

## What does the Fourier transform of a $gH$ coset function look like?

Assume that $\rho$ is $G > H > 1$ adapted.

$$\widehat{f}(\rho) = \sum_{g \in G/H} \rho(g)\, f(gH) \sum_{h \in H} \rho(h)$$

$$\sum_{h \in H} \rho(h) = \sum_{h \in H} \bigoplus_{\rho'} \rho'(h) \quad \text{but} \quad \sum_{h \in H} \rho'(h) = 0$$

unless $\rho'$ is the trivial representation.

Only those columns of $\widehat{f}(\rho)$ are non-zero where we have trivial $\rho'$.